

Entanglement and extreme spin squeezing of unpolarized states

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We present an optimal set of criteria detecting the depth of entanglement in macroscopic systems of general spin- j particles using the variance and second moments of the collective spin components. The class of states detected goes beyond traditional spin-squeezed states by including Dicke states and other unpolarized states. The criteria derived are easy to evaluate numerically even for systems composed of a very large number of particles and outperform past approaches, especially in practical situations where noise is present. We also derive analytic lower bounds based on the linearization of our criteria, which make it possible to define spin-squeezing parameters for Dicke states. As a by-product, we obtain also an analytic lower bound to the condition derived in [A. S. Sørensen and K. Mølmer, *Phys. Rev. Lett.* **86**, 4431 (2001)]. We also extend the results to systems with fluctuating number of particles.

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With an interest towards fundamental questions in quantum physics, as well as applications, larger and larger entangled quantum systems have been realized with photons, trapped ions, and cold atoms [1–11]. Entanglement is needed for certain quantum information processing tasks [12, 13], and it is also necessary to reach the maximum sensitivity in a wide range of interferometric schemes in quantum metrology [14]. Hence, the verification of the presence of entanglement is a crucial but exceedingly challenging task, especially in an ensemble of many, say $10^3 - 10^{12}$ particles [5–11]. Moreover, in such experiments it is not sufficient to claim that “the state is entangled”, we need also to know how entangled the system is. Hence, quantifying entanglement in large ensembles has recently been at the center of attention. In several experiments the *entanglement depth* was determined, i.e., the minimal number of mutually entangled particles consistent with the measurement data, reaching to the thousands [7–11].

In the many-particle case, it is typically impossible to address the particles individually, and only collective quantities can be measured. In this context, one of the most successful approaches to detect entanglement is based on the criterion [15]

$$\xi_s^2 := N \frac{(\Delta J_x)^2}{\langle J_y \rangle^2 + \langle J_z \rangle^2} \geq 1, \quad (1)$$

where N is the particle number, $J_l = \sum_{n=1}^N j_l^{(n)}$ for $l = x, y, z$ are the collective spin components, and $j_l^{(n)}$ are single particle spin components acting on the n th particle. The criterion (1) is best suited for states with a large collective spin in the (\hat{y}, \hat{z}) -plane and a small variance $(\Delta J_x)^2$ in the orthogonal direction. Every multi-qubit state that violates Eq. (1) must be entangled [15] and has been also called *spin squeezed* in the context of

metrology [16, 17], due to the fact that the variance of a spin component is reduced below what can be achieved with fully polarized spin-coherent states.

As a generalization of Eq. (1), a criterion has also been derived by Sørensen and Mølmer [18], allowing to detect the depth of entanglement of spin-squeezed states by violating the condition

$$(\Delta J_x)^2 \geq N j F_{kj} \left(\frac{\langle J_z \rangle}{N j} \right), \quad (2)$$

based on the function

$$F_J(X) := \frac{1}{J} \min_{\langle L_z \rangle = X} (\Delta L_x)^2, \quad (3)$$

where L_l are the spin components of a single particle with spin J . As explained in Ref. [18], the functions F_J with $J = kj$ provide a boundary region for states with an entanglement depth of at most k in the $(\langle J_z \rangle, (\Delta J_x)^2)$ -plane and thus every multi-spin- j state that violates Eq. (2) must have a depth of entanglement of $(k + 1)$ or larger. Spin squeezing has been demonstrated in many experiments, from cold atoms [7, 19–26] to trapped ions [27], magnetic systems [28, 29] and photons [30].

Recently, the concept of spin squeezing has also been extended to include states that are not fully polarized, such as singlet states, planar squeezed states and Dicke states [8, 31–34]. The last in particular, which are another important class of multipartite entangled states, are attracting increasing attention and are being produced in experiments with photons [35, 36] and Bose-Einstein condensates [8, 24, 37]. Suitable criteria to detect the depth of entanglement of Dicke states have also been derived [8, 38, 39], but either they are limited to spin-1/2 particles or they are not optimal in the sense that they do not detect all states that could be detected based on the measured quantities.

In this paper we present a very general condition that outperforms all the entanglement criteria mentioned above, since (i) it detects multipartite rather than only bipartite entanglement, (ii) is applicable to spin- j systems, for any j , (iii) works both for spin-squeezed states and Dicke states, and (iv) is an optimal condition. Such a criterion can be applied immediately in experiments with Dicke states in spinor condensates [40].

Summary of the main results.—We show that the condition

$$(\Delta J_x)^2 \geq NjF_{kj} \left(\sqrt{\frac{\langle J_y^2 + J_z^2 \rangle - Nj(kj+1)}{N(N-k)j^2}} \right) \quad (4)$$

holds for states with an entanglement depth at most k of an ensemble of N spin- j particles. We also analyze the performance of our condition compared to other criteria in the literature.

In general, the function $F_J(X)$ appearing on the right-hand side of Eq. (4) has to be evaluated numerically. However due to its convexity properties we can bound it from below with the lowest nontrivial order of its Taylor expansion, yielding spin-squeezing parameters similar to the one defined in Eq. (1). While states saturating Eq. (4) determine a curve in the $(\langle J_y^2 + J_z^2 \rangle, (\Delta J_x)^2)$ -plane, these analytic conditions correspond to tangents to this curve. Hence, we will call them linear criteria in the paper. A family of such conditions for states with an entanglement depth k or smaller (such that kj is integer) is the inequality

$$\xi^2 := (kj+1) \frac{2(N-k)j(\Delta J_x)^2}{\langle J_y^2 + J_z^2 \rangle - Nj(kj+1)} \geq 1. \quad (5)$$

A similar condition can be obtained from the Sørensen-Mølmer criteria (2) as

$$\xi_{\text{SM}}^2 := (kj+1) \frac{2Nj(\Delta J_x)^2}{\langle J_y^2 \rangle + \langle J_z^2 \rangle} \geq 1. \quad (6)$$

A direct comparison between ξ^2 and ξ_{SM}^2 shows that Eq. (5) is more suitable for detecting the depth of entanglement of unpolarized states, such as Dicke states, and also that, compared to Eq. (6), it takes advantage of the anti-squeezing on $(\Delta J_y)^2$ for fully polarized states. Note also the similarity of Eq. (6) and Eq. (1). All these criteria are also generalized to the fluctuating- N case, following Ref. [41].

Optimal nonlinear criteria.—We distinguish various levels of multipartite entanglement based on the following definitions. A quantum state is k -producible if it can be written as $\rho = \sum_i p_i \rho_i^{(1)} \otimes \rho_i^{(2)} \otimes \dots$ for a suitable choice of states $\rho_i^{(l)}$ of $k_l \leq k$ particles, where p_i are probabilities. Clearly, 1-producible states are separable states. A state that is not k -producible is called $(k+1)$ -entangled. The entanglement depth is $k+1$ whenever the

state is $(k+1)$ -producible but not k -producible [18, 42]. For clarity we introduce the notation $G_J : X \mapsto F_J(\sqrt{X})$.

Observation 1.—The inequality in Eq. (4) holds for all k -producible states of N spin- j particles. The condition can be used if $\langle J_y^2 + J_z^2 \rangle \geq Nj(kj+1)$, while otherwise $(\Delta J_x)^2 = 0$ can be achieved. Thus, every state of N spin- j particles that violates Eq. (4) must be $(k+1)$ -entangled.

Proof.—The key argument of the proof is that for pure k -producible states of N spin- j particles

$$\sqrt{\frac{\langle J_y^2 + J_z^2 \rangle - Nj(kj+1)}{N(N-k)j^2}} \leq \frac{\sqrt{\langle J_y \rangle^2 + \langle J_z \rangle^2}}{Nj} \quad (7)$$

holds. Then, for pure k -producible states $(\Delta J_x)^2 \geq NjF_{kj}(\text{RHS}) \geq NjF_{kj}(\text{LHS})$ follows from the properties of the $F_J(X)$, where we used the notation RHS and LHS for the left-hand side and right-hand side of Eq. (7), respectively. The first inequality originates from the result of Sørensen-Mølmer, Eq. (2), while the second follows from monotonicity of F_J . Finally, the inequality can be extended to mixed k -producible states since the functions G_J are convex and the argument is linear under mixing the state. This fact is discussed together with the proof of Eq. (7) and the tightness of Eq. (4), in the supplemental material. \square

These criteria are especially suited to detect states for which $\langle J_y^2 + J_z^2 \rangle$ is large and $(\Delta J_x)^2$ is small. A paradigmatic example for this is the unpolarized Dicke state in the x -basis, $\rho_{\text{Dicke}} = |J = Nj, m_x = 0\rangle\langle J = Nj, m_x = 0|$ which is detected as N -entangled.

Numerical computation of G_J .—The value of $G_J(X)$ can be obtained from the ground states of

$$H_\lambda = L_x^2 - \lambda L_z, \quad (8)$$

which are spin-squeezed states [18], and is feasible even for J of the order of thousands. When determining $G_J(X)$ for a given X , we need to carry out an optimization over λ . However, to study $G_J(X)$ for some interval of X , the explicit optimization can be avoided, which makes it possible to carry out calculations for very large systems. We just need to calculate the ground states $|\phi_\lambda\rangle$ of Eq. (8) for a sufficiently wide interval of λ , and then the points of the curve are obtained as $X = \frac{1}{J^2} \langle L_z \rangle_{\phi_\lambda}^2$ and $G_J(X) = \frac{1}{J} \langle L_x^2 \rangle_{\phi_\lambda}$. The result is shown in Fig. 1. Correspondingly, the boundary for k -producible states in the $(\langle J_y^2 + J_z^2 \rangle, (\Delta J_x)^2)$ -plane is parametrized by (cf. Fig. 2)

$$\begin{aligned} \langle J_y^2 + J_z^2 \rangle_\lambda &= \frac{N(N-k)j^2}{k^2j^2} \langle L_z \rangle_{\phi_\lambda}^2 + Nj(kj+1), \\ (\Delta J_x)_\lambda^2 &= \frac{N}{k} (\Delta L_x)_{\phi_\lambda}^2. \end{aligned} \quad (9)$$

We mention that for $J = 1$ we have $G_1(X) = \frac{1}{2}(1 - \sqrt{1-X})$, i.e., the function on the right hand side of

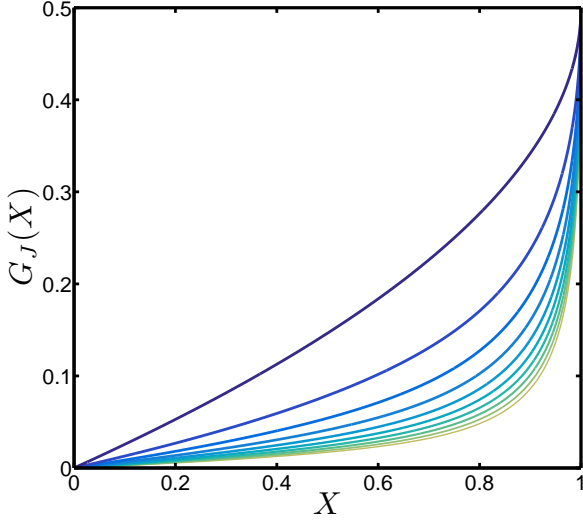


FIG. 1. (color online). The function $G_J(X)$ for (left to right) $J = 1, 3, 5, \dots, 19$.

the criteria can be obtained analytically. Substituting $F_1(X) = G_1(X^2)$ into Eq. (4), we can obtain an analytic 2-producibility condition for qubits and an analytic separability condition for qutrits. For higher J , $G_J(X)$ is not known analytically.

Based on uncertainty relations of angular momentum operators a lower bound on $G_J(X)$ can be obtained as

$$\tilde{G}_J(X) = \frac{1}{2} \left[(J+1) - JX - \sqrt{(J+1 - JX)^2 - X} \right], \quad (10)$$

which is not tight for small J and small X , but becomes optimal for large J and X close to 1, see Ref. [18]. Complementary to that approximation, in the following we derive a lower bound on $G_J(X)$ that is optimal for $X \approx 0$ and improves \tilde{G}_J at small X by a factor of 2.

Linear analytic criteria.—For integer J , we can compute the first terms of the Taylor expansion of $G_J(X)$ about $X = 0$ and, using its convexity, obtain the bound $G_{kj}(X) \geq (G_{kj}(0) + XG'_{kj}(0))$. In other words, we can compute the tangent to the k -producibility boundaries, in the vicinity of their intersection point with the horizontal axis. This expansion can be done by employing the perturbation series for H_λ in powers of the parameter $\lambda \ll 1$, since $X = 0$ also corresponds to $\lambda = 0$. In particular we have that $|\phi_0\rangle$ is the eigenstate of L_x with eigenvalue zero and $|\phi_1\rangle = \sum_{m \neq 0} c_m |m\rangle_x$ with $c_m = -\langle m | L_z | 0 \rangle_x / m^2$ and $|m\rangle_x$ being the eigenstates of L_x with eigenvalue m . Thus, the ground state of H_λ is

$$|\phi_\lambda\rangle = |0\rangle_x - i\lambda \frac{\sqrt{J(J+1)}}{2} (|1\rangle_x - |-1\rangle_x) + O(\lambda^2), \quad (11)$$

from which we obtain $X(\lambda) \approx \lambda^2 (J+1)^2$ and $G_J(\lambda) \approx$

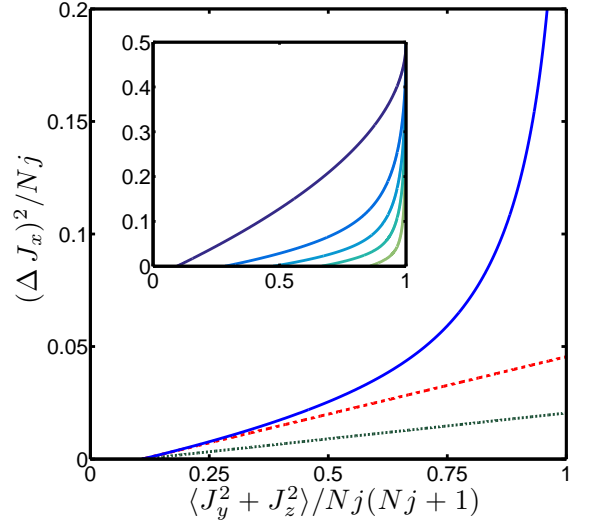


FIG. 2. (color online). 20-producibility criteria for $N = 200$ qubits. (solid) The optimal boundary, Eq. (4). The dashed and dotted lines are Eq. (15), i.e., the tangent to the curve, and the criterion (14) respectively. In the inset, curves are shown for k -producibility for $N = 20$ spin-1 particles, for $k = 1, 5, 9, 13, 17$.

$\frac{1}{2}\lambda^2(J+1)$ [43]. Hence, we arrive at

$$G_J(X) \geq \frac{X}{2(J+1)}, \quad (12)$$

by employing the chain rule for $\frac{dG_J(X(\lambda))}{d\lambda}$. Note that taking the derivative in Eq. (10) one obtains $G_J(X) \geq \tilde{G}_J(X) \geq X\tilde{G}'_J(0) = \frac{X}{4(J+1)}$, which underestimates Eq. (12) by a factor of 2. Based on this, we can derive analytic criteria that become optimal close to the point $(\Delta J_x)^2 = 0$.

Observation 2.—The criteria in Eqs. (5,6) hold for all k -producible states of N spin- j particles such that $J = kj$ is an integer number. Every state of N spin- j particles that violates them must be $(k+1)$ -entangled.

Proof.—From Eq. (12) with $J = kj$ we can bound from below the criteria (4) with Eq. (5) by substituting $X = [\langle J_y^2 + J_z^2 \rangle - Nj(kj+1)] / [N(N-kj)^2]$. Analogously, by rewriting Eq. (2) in terms of G_{kj} and using the bound (12) with $X = \langle J_z \rangle^2 / N^2 j^2$ we obtain (6). \square

See Fig. 2 for a plot of Eq. (5) as the tangent to the boundary of 20-producibility for $N = 200$ spin $j = \frac{1}{2}$ particles in the $(\langle J_y^2 + J_z^2 \rangle, (\Delta J_x)^2)$ -plane.

Comparison with similar criteria.—Due to the monotonicity of F_J , for comparing Eq. (2) and Eq. (4) it suffices to compare the arguments of the functions.

Observation 3.—Whenever the condition

$$\frac{(\Delta J_y)^2 + (\Delta J_z)^2}{Nj} > kj \left(1 - \frac{\langle J_y \rangle^2 + \langle J_z \rangle^2}{N^2 j^2} \right) + 1 \quad (13)$$

holds, then Eq. (4) is strictly stronger than the Sørensen–Mølmer criteria (2).

Proof.—(a) Since $F_J(X) \geq F_J(Y)$ for $X \geq Y$ we have that Eq. (4) implies Eq. (2) whenever Eq. (13) holds. Then, let us consider states $\rho_{\text{Dicke},p} = (1-p)\rho_{\text{Dicke}} + p\frac{1}{(2j+1)^N}$ close to the unpolarized Dicke state, for which $\langle J_y \rangle_{\text{Dicke},p}^2 + \langle J_z \rangle_{\text{Dicke},p}^2 = 0$ holds identically for all p . From the linear criteria, Eq. (5), we obtain the bound $p < \frac{3(N-k)j}{2j(j+1)(kj+1)(N-k)-2(j+i)+3(Nj+1)}$ for the degree of white noise such that $\rho_{\text{Dicke},p}$ is detected with Eq. (4), but not with Eq. (2), since in that case the right hand side is zero for all p . \square

In practice, Eq. (13) holding is the most common situation, especially when noise is present that increases the variances on the left hand side. As the two extremal (and practically most important) examples we mention that it holds for states such that $(\Delta J_y)^2 + (\Delta J_z)^2 \gg Nkj^2$ (i.e., states close to Dicke states) and for states such that $(\Delta J_y)^2 + (\Delta J_z)^2 \sim O(Nj)$ while $\langle J_y \rangle^2 + \langle J_z \rangle^2 \sim O(N^2j^2)$ (i.e., typical fully polarized states). Note also that from Eq. (13) it is clear that our inequality takes advantage of the anti-squeezing of the variance in the orthogonal direction for fully polarized spin squeezed states. For example, simple numerics show that for spin-squeezed states obtained as ground states of $H_\mu = J_z^2 - \mu J_x$ the condition (13) is fulfilled for $N = 1000$ spin- $\frac{1}{2}$ particles for all μ , if white noise is added to the state with a small noise fraction $p = 0.01$.

Furthermore, we compare our criteria with another important set of conditions that are designed to detect the entanglement depth near unpolarized Dicke states. These are linear criteria derived by Duan in Ref. [38], stating that

$$N(k+2)(\Delta J_x)^2 \geq \langle J_y^2 + J_z^2 \rangle - \frac{N}{4}(k+2) \quad (14)$$

holds for all k -producible states of N spin- $\frac{1}{2}$ particles. Any state that violates Eq. (14) is detected as $(k+1)$ -entangled. In this case, we can compare it with the linear criteria (5), specialized for qubit-systems, i.e., with $j = \frac{1}{2}$

$$\frac{(N-k)}{2}(k+2)(\Delta J_x)^2 \geq \langle J_y^2 + J_z^2 \rangle - \frac{N}{4}(k+2). \quad (15)$$

It is easy to see that a violation of Eq. (14) implies a violation of Eq. (15) and thus that our conditions detect strictly more states (see also Fig. 2).

Finally, we note that Eq. (4) with $j = 1/2$ is similar to the criterion for spin-1/2 particles used in the experiment described in Ref. [8]. A key difference is that in Eq. (4), in the denominator of the fraction, the term $N(N-k)j^2 = N(N-k)/4$ appears, while in the formula of Ref. [8] there is the term $N^2/4$. The difference between the two criteria is the largest when we examine highly entangled Dicke states or spin-squeezed states, and in the argument of $F_{kj}(X)$ we have a value close to $X = 1$.

In the vicinity of this point, the derivative of $F_{kj}(X)$ is very large, hence some improvement in the argument of $F_{kj}(X)$ makes the bound on the right-hand side of Eq. (4) significantly higher. As a consequence, the criterion (4) is optimal for noisy Dicke states of many particles even in $k \sim N$ case, while the criterion of Ref. [8] is optimal only when $k \ll N$, and it does not detect the Dicke state as N -entangled.

Extension to fluctuating number of particles.—

For macroscopic ensembles of particles, e.g., $N \sim 10^6$, the total number is not under perfect control and is impossible to collect statistical data for fixed N . This issue has been studied by Hyllus *et al.* [41], who suitably generalized the definition of entanglement depth and, by exploiting the concavity of variance, also all the spin-squeezing criteria to the case of fluctuating number of particles. Using similar methods, let us consider density matrices $\rho = \sum_N Q_N \rho_N$, where ρ_N are the density matrices in some fixed- N subspace and Q_N are probabilities. Also, let us consider the global collective spin operators $J_I = \sum_N J_{I,N}$.

Then, the same spin-squeezing criteria, e.g., Eq. (2), hold in the fluctuating N case simply under the substitution $N \rightarrow \langle N \rangle$, i.e., as [41]

$$(\Delta J_x)^2 \geq \langle N \rangle j F_{kj} \left(\frac{\langle J_z \rangle}{\langle N \rangle j} \right). \quad (16)$$

In our case, we can apply a similar reasoning.

Observation 4.—All k -producible states in the Hilbert space $\mathcal{H} = \bigoplus \mathcal{H}_N$ must satisfy the following inequality

$$(\Delta J_x)^2 \geq \langle N \rangle j G_{kj} \left(\frac{\langle W \rangle}{\langle N \rangle j} \right), \quad (17)$$

where

$$W = \sum_N (Nj - kj)^{-1} [J_{y,N}^2 + J_{z,N}^2 - Nj(kj + 1)\mathbb{1}_N], \quad (18)$$

and $\langle W \rangle \geq 0$ is required.

Proof.—Given a state $\rho = \sum_N Q_N \rho_N$ we have that (17) follows from $(\Delta J_k)^2 \geq \sum_N Q_N (\Delta J_{k,N})^2$ and Jensen's inequality $\frac{1}{\sum_N Q_N N} \sum_N Q_N N G_{kj}(X_N) \geq G_{kj}(\frac{\sum_N Q_N N X_N}{\sum_N Q_N N})$ with $X_N = \frac{\langle W \rangle_{\rho_N}}{N}$ and $\langle N \rangle = \sum_N Q_N N$. \square

Note that the operator W defined in Eq. (18) is simply a direct sum of $J_{y,N}^2 + J_{z,N}^2$ over all fixed- N subspaces, normalized with a factor $1/(Nj - kj)$. Thus, to apply our condition in experiments with fluctuating number of particles, one needs to measure the spin operators and the particle number jointly at each shot, and then average over an ensemble to compute $\langle W \rangle$.

Conclusions.—We derived a set of criteria to determine the depth of entanglement of spin-squeezed states

and unpolarized Dicke states, extending and completing both the results of Refs. [8, 18]. These generalized spin-squeezing conditions are valid even for an ensemble of spin- j particles with $j > \frac{1}{2}$, which is very useful, since most experiments are done with particles with a higher spin, e.g., with spin-1 ^{87}Rb atoms. Since theory is mostly available for the spin- $\frac{1}{2}$ case, pseudo spin- $\frac{1}{2}$ particles are created artificially such that only two of the levels are populated. While the spin-squeezing approach to entanglement detection is already widely used in such systems [7, 8, 19–26, 33], our criteria make it possible to study spin-squeezing in fundamentally new experiments. A clear advantage of using the physical spin is that it is typically much easier to manipulate than the pseudo spin- $\frac{1}{2}$ particles. In future, it would be interesting to clarify the relation between generalized spin squeezing and metrological usefulness [44–49], and also compare our results with the complete set of spin-squeezing criteria of Ref. [50], that contain one additional collective observable, related to single-spin average squeezing.

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- [1] C. Schwemmer, G. Tóth, A. Niggelbaum, T. Moroder, D. Gross, O. Gühne, and H. Weinfurter, *Phys. Rev. Lett.* **113**, 040503 (2014).
 - [2] W.-B. Gao, C.-Y. Lu, X.-C. Yao, P. Xu, O. Gühne, A. Goebel, Y.-A. Chen, C.-Z. Peng, Z.-B. Chen, and J.-W. Pan, *Nat. Phys.* **6**, 331 (2010).
 - [3] D. Leibfried, M. Barrett, T. Schaetz, J. Britton, J. Chiaverini, W. Itano, J. Jost, C. Langer, and D. Wineland, *Science* **304**, 1476 (2004).
 - [4] Y.-F. Huang, B.-H. Liu, L. Peng, Y.-H. Li, L. Li, C.-F. Li, and G.-C. Guo, *Nat. Commun.* **2**, 546 (2011).
 - [5] J. Appel, P. J. Windpassinger, D. Oblak, U. B. Hoff, N. Kjærgaard, and E. S. Polzik, *PNAS* **106**, 10960 (2009).
 - [6] R. J. Sewell, M. Koschorreck, M. Napolitano, B. Dubost, N. Behbood, and M. W. Mitchell, *Phys. Rev. Lett.* **109**, 253605 (2012).
 - [7] C. Gross, T. Zibold, E. Nicklas, J. Esteve, and M. K. Oberthaler, *Nature* **464**, 1165 (2010).
 - [8] B. Lücke, J. Peise, G. Vitagliano, J. Arlt, L. Santos, G. Tóth, and C. Klempt, *Phys. Rev. Lett.* **112**, 155304 (2014).
 - [9] O. Hosten, N. J. Engelsen, R. Krishnakumar, and M. A. Kasevich, *Nature* **529**, 505 (2016).
 - [10] R. McConnell, H. Zhang, J. Hu, S. Čuk, and V. Vuletić, *Nature (London)* **519**, 439 (2015).
 - [11] F. Haas, J. Volz, R. Gehr, J. Reichel, and J. Esteve, *Science* **344**, 180 (2014).
 - [12] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, *Rev. Mod. Phys.* **81**, 865 (2009).
 - [13] O. Gühne and G. Tóth, *Phys. Rep.* **474**, 1 (2009).
 - [14] L. Pezzé and A. Smerzi, *Phys. Rev. Lett.* **102**, 100401 (2009).
 - [15] A. Sørensen, L.-M. Duan, J. Cirac, and P. Zoller, *Nature* **409**, 63 (2001).
 - [16] M. Kitagawa and M. Ueda, *Phys. Rev. A* **47**, 5138 (1993).
 - [17] D. J. Wineland, J. J. Bollinger, W. M. Itano, and D. J. Heinzen, *Phys. Rev. A* **50**, 67 (1994).
 - [18] A. S. Sørensen and K. Mølmer, *Phys. Rev. Lett.* **86**, 4431 (2001).
 - [19] J. Hald, J. L. Sørensen, C. Schori, and E. S. Polzik, *Phys. Rev. Lett.* **83**, 1319 (1999).
 - [20] T. Fernholz, H. Krauter, K. Jensen, J. F. Sherson, A. S. Sørensen, and E. S. Polzik, *Phys. Rev. Lett.* **101**, 073601 (2008).
 - [21] M. F. Riedel, P. Böhi, Y. Li, T. W. Hänsch, A. Sinatra, and P. Treutlein, *Nature* **464**, 1170 (2010).
 - [22] C. Orzel, A. K. Tuchman, M. L. Fenselau, M. Yasuda, and M. A. Kasevich, *Science* **291**, 2386 (2001).
 - [23] J. Esteve, C. Gross, A. Weller, S. Giovanazzi, and M. Oberthaler, *Nature* **455**, 1216 (2008).
 - [24] C. Hamley, C. Gerving, T. Hoang, E. Bookjans, and M. Chapman, *Nat. Phys.* **8**, 305 (2012).
 - [25] J. G. Bohnet, K. C. Cox, M. A. Norcia, J. M. Weiner, Z. Chen, and J. K. Thompson, *Nat. Photon* **8**, 731 (2014).
 - [26] K. C. Cox, G. P. Greve, J. M. Weiner, and J. K. Thompson, *Phys. Rev. Lett.* **116**, 093602 (2016).
 - [27] V. Meyer, M. A. Rowe, D. Kielpinski, C. A. Sackett, W. M. Itano, C. Monroe, and D. J. Wineland, *Phys. Rev. Lett.* **86**, 5870 (2001).
 - [28] M. S. Reis, S. Soriano, A. M. dos Santos, B. C. Sales, D. O. Soares-Pinto, and P. Brandão, *Europhys. Lett.* **100**, 50001 (2012).
 - [29] R. Auccaise, A. G. Araujo-Ferreira, R. S. Sarthour, I. S. Oliveira, T. J. Bonagamba, and I. Roditi, *Phys. Rev. Lett.* **114**, 043604 (2015).
 - [30] M. W. Mitchell and F. A. Beduini, *New J. Phys.* **16**, 073027 (2014).
 - [31] G. Tóth, C. Knapp, O. Gühne, and H. J. Briegel, *Phys. Rev. Lett.* **99**, 250405 (2007).
 - [32] Q. Y. He, S.-G. Peng, P. D. Drummond, and M. D. Reid, *Phys. Rev. A* **84**, 022107 (2011).
 - [33] N. Behbood, F. M. Ciurana, G. Colangelo, M. Napolitano, G. Tóth, R. J. Sewell, and M. W. Mitchell, *Phys. Rev. Lett.* **113**, 093601 (2014).
 - [34] G. Vitagliano, I. Apellaniz, I. L. Egusquiza, and G. Tóth, *Phys. Rev. A* **89**, 032307 (2014).
 - [35] W. Wieczorek, R. Krischek, N. Kiesel, P. Michelberger, G. Tóth, and H. Weinfurter, *Phys. Rev. Lett.* **103**, 020504 (2009).
 - [36] R. Prevedel, G. Cronenberg, M. S. Tame, M. Paternostro, P. Walther, M. S. Kim, and A. Zeilinger,

- Phys. Rev. Lett. **103**, 020503 (2009).
- [37] B. Lücke, M. Scherer, J. Kruse, L. Pezzé, F. Deuretzbacher, P. Hyllus, J. Peise, W. Ertmer, J. Arlt, L. Santos, A. Smerzi, and C. Klempt, *Science* **334**, 773 (2011).
- [38] L.-M. Duan, *Phys. Rev. Lett.* **107**, 180502 (2011).
- [39] Z. Zhang and L.-M. Duan, *Phys. Rev. Lett.* **111**, 180401 (2013).
- [40] T. M. Hoang, M. Anquez, M. J. Boguslawski, H. M. Bharath, B. A. Robbins, and M. S. Chapman, *arXiv:1512.06766* (2015).
- [41] P. Hyllus, L. Pezzé, A. Smerzi, and G. Tóth, *Phys. Rev. A* **86**, 012337 (2012).
- [42] O. Gühne, G. Tóth, and H. J. Briegel, *New J. Phys.* **7**, 229 (2005).
- [43] $O(x)$ is the usual Landau notation used to describe the asymptotic behavior of a quantity for large x .
- [44] P. Hyllus, W. Laskowski, R. Krischek, C. Schwemmer, W. Wieczorek, H. Weinfurter, L. Pezzé, and A. Smerzi, *Phys. Rev. A* **85**, 022321 (2012).
- [45] G. Tóth, *Phys. Rev. A* **85**, 022322 (2012).
- [46] Z. Zhang and L. M. Duan, *New J. Phys.* **16**, 103037 (2014).
- [47] G. Tóth and I. Apellaniz, *J. Phys. A: Math. Theor.* **47**, 424006 (2014).
- [48] I. Apellaniz, B. Lücke, J. Peise, C. Klempt, and G. Tóth, *New J. Phys.* **17**, 083027 (2015).
- [49] I. Apellaniz, M. Kleinmann, O. Gühne, and G. Tóth, *arXiv:1511.05203*.
- [50] G. Vitagliano, P. Hyllus, I. L. Egusquiza, and G. Tóth, *Phys. Rev. Lett.* **107**, 240502 (2011).

SUPPLEMENTAL MATERIAL

Proof. of Eq. (7)—To prove Eq. (7), let us consider the expression $(\Delta J_y)^2 + (\Delta J_z)^2$ on pure k -producible states $\rho = \sum_i p_i \rho_i^{(1)} \otimes \dots \otimes \rho_i^{(N_k)}$, where N_k stands for the total number of groups. We have, due to the additivity of the variance for tensor products

$$\begin{aligned} (\Delta J_y)^2 + (\Delta J_z)^2 &= \sum_l \left[(\Delta j_y^{(l)})^2 + (\Delta j_z^{(l)})^2 \right] \\ &\leq \sum_l \left[k_l j (k_l j + 1) - \langle (j_x^{(l)})^2 \rangle - \langle j_y^{(l)} \rangle^2 - \langle j_z^{(l)} \rangle^2 \right], \end{aligned} \quad (\text{S1})$$

where the superscript (l) indicates the l th group, that is composed of k_l particles. Note that we also have $\frac{N}{k} \leq N_k \leq N$ and $\sum_{l=1}^{N_k} k_l = N$. Then, since $\langle (j_x^{(k_l)})^2 \rangle \geq 0$ we can neglect it and obtain

$$\begin{aligned} \langle J_y^2 + J_z^2 \rangle &\leq \langle J_y \rangle^2 + \langle J_z \rangle^2 \\ &+ \sum_l k_l j \left[(k_l j + 1) - k_l j \frac{(\langle j_y^{(l)} \rangle^2 + \langle j_z^{(l)} \rangle^2)}{k_l^2 j^2} \right], \end{aligned} \quad (\text{S2})$$

that can be bounded as

$$\begin{aligned} \langle J_y^2 + J_z^2 \rangle &\leq \langle J_y \rangle^2 + \langle J_z \rangle^2 + Nj \\ &+ \sum_l k_l j \left[k_j \left(1 - \frac{\langle j_y^{(l)} \rangle^2 + \langle j_z^{(l)} \rangle^2}{k_l^2 j^2} \right) \right], \end{aligned} \quad (\text{S3})$$

due to the fact that $\max_l k_l = k$ and that the expression inside the round brackets in Eq. (S3) is positive. Furthermore, using Jensen's inequality in the form

$$-\sum_l k_l f_l^2 \leq -\frac{1}{N} \left(\sum_l k_l f_l \right)^2, \quad \sum_l k_l = N, \quad (\text{S4})$$

with $f_l = \frac{\langle j_m^{(l)} \rangle}{k_l}$ for $m = x, y, z$ we obtain

$$\begin{aligned} \langle J_y^2 + J_z^2 \rangle - Nj(kj + 1) \\ \leq \left(1 - \frac{k}{N} \right) (\langle J_y \rangle^2 + \langle J_z \rangle^2). \end{aligned} \quad (\text{S5})$$

Tightness of Eq. (7)—In deriving Eq. (S5) we neglected the contribution of $\sum_l \langle (j_x^{(l)})^2 \rangle := \mathcal{X}$ which is a positive collective operator related to single particle spin squeezing. Also, for pure product states in general we have $-\mathcal{X} \leq -(\Delta J_x)^2$ with equality holding in the case $\langle J_x \rangle = 0$. For simplicity we omitted such \mathcal{X} from Eq. (S5), that is thus not completely tight. However, it is still tight in the large N limit and in all practical cases, i.e., for experiments with Dicke states or fully polarized states for which $(\Delta J_x)^2 = O(N)$ and $\langle J_y^2 + J_z^2 \rangle = O(N^2)$. As a consequence, the criterion (4) is optimal in the large N limit.

Properties of F_J .—The functions $F_J(X)$ can be obtained from the optimal states ρ for the problem defined in Eq. (3), i.e., the states that minimize $(\Delta L_x)^2$ for a given $\langle L_z \rangle$. These can be found as ground states of $H = (L_x - \lambda_x)^2 - \lambda L_z$ with the two parameters λ_x and λ . For each given λ we can take the minimum ground state energy varying over λ_x and for an integer spin J such minimum is attained for $\lambda_x = 0$. Thus, for integer J , $F_J(X)$ is giving the minimal $\langle J_x^2 \rangle$ for a given value of $\langle J_z \rangle$. Since the set of physical states is convex, the set of points in the $(\langle J_z \rangle, \langle J_x^2 \rangle)$ -space corresponding to physical states is also convex. Hence, $F_J(X)$ is also a convex function and in particular its derivative $\lambda(X)$ is monotonously increasing with X . Note that in Ref. [18] a different proof was presented. In principle, the derivative $F'_J(X)$ can be computed by numerical derivation of $F_J(X)$. However, it is much simpler to obtain $F'_J(X)$ for some range of X by plotting $(\frac{1}{J} \langle L_z \rangle_{\phi_\lambda}, \lambda)$ for some range of λ [18]. In other words, for $X = \frac{1}{J} \langle L_z \rangle_{\phi_\lambda}$ the derivative is $F'_J(X) = \lambda$.

To show that also $G_J(X)$ is convex we observe that $G'_J(X) = \frac{1}{2\sqrt{X}} F'_J(\sqrt{X})$ is a monotonously increasing function of X . We evaluate numerically the derivative $G'_J(X)$ by plotting $(\frac{1}{J^2} \langle L_z \rangle_{\phi_\lambda}^2, \frac{J}{2\langle L_z \rangle_{\phi_\lambda}} \lambda)$ for a wide range of λ , cf. Fig. S1, and see explicitly its monotonicity. More in general one can check whether or not $F_J(X^{\frac{1}{\alpha}})$ is convex for any exponent α . It can then be observed numerically (not shown) that $F_J(X^{\frac{1}{\alpha}})$ is not convex for any $\alpha > 2$.

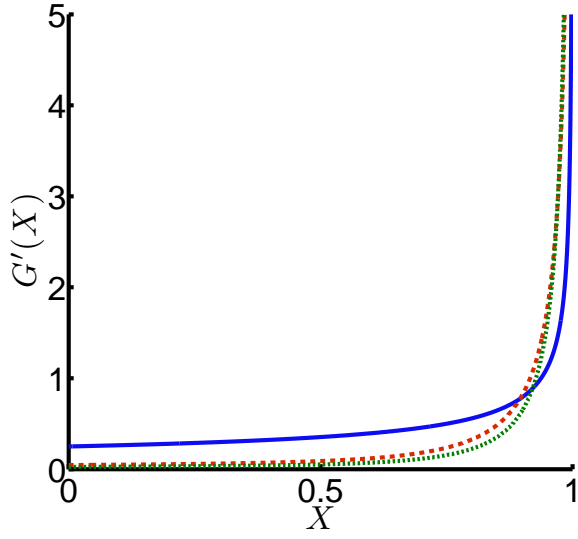


FIG. S1. (color online). The derivative $G'(X) = \frac{J}{2\langle L_z \rangle_{\phi_\lambda}} \lambda$ as a function of $X = \frac{1}{J^2} \langle L_z \rangle_{\phi_\lambda}^2$ for $J = 1$ (continuous line), $J = 10$ (dashed) and $J = 19$ (dotted).